Analysis and Simulation of Group Dynamics based on Heider's Balance Theory and a Finite Markov Chain

Tatsuya Nomura

Department of Media Informatics, Ryukoku University
1–5, Yokotani, Setaohe–cho, Otsu, Shiga 520–2194, Japan
E-mail: nomura@rins.ryukoku.ac.jp

Abstract. Heider's balance theory is one of theories on micro characteristics of triad relations in social psychology. However, it has not sufficiently been discussed what relations there are between group dynamics and this micro characteristic, that is, what situations appear in convergence of the group dynamics based on balance of individual triads. This paper proposes a formalization of this group dynamics as a finite Markov chain, mathematically analyzes absorbing states of this Markov chain, and verifies their characteristics based on computer simulations. Moreover, it considers influence of a person fixing relations to others through the process in this Markov chain.

1 Introduction

As one of theories on micro characteristics of individuals in social psychology, balance theory proposed by F. Heider [5] states a psychological stability of an individual included in a triad relation. In this theory, a person (P), another person (O), an object or the third person (X), and relations from P to O, from O to X, and from P to X construct a system (called POX system). These relations have either $+$ or $-$ value corresponding to the fact that the person likes or dislikes the object respectively. Heider's theory argues that a POX system is balanced if and only if the product of the signs on these three relations is $+$, and if the system is not balanced P changes one of the relations to O and X so that the POX system becomes balanced. As shown in Fig. 1, if the system is not balanced, then P inverts either the sign of $P \rightarrow O$ or that of $P \rightarrow X$ to balance the POX system.

Although the original balance theory is limited to triad relations, its extension to groups consisting of more than three persons have been proposed [1, 3, 7]. The concept of balance in social networks as graphs has been applied in several fields of social science [7]. These studies of balance in social networks focus on network structures of balanced situations based on graph theory. However, it has not sufficiently discussed what characteristics balanced graph structures have in the sense of micro–macro dynamics, more concretely, whether balanced situations really appear and what graph structures actually appear in large groups as a

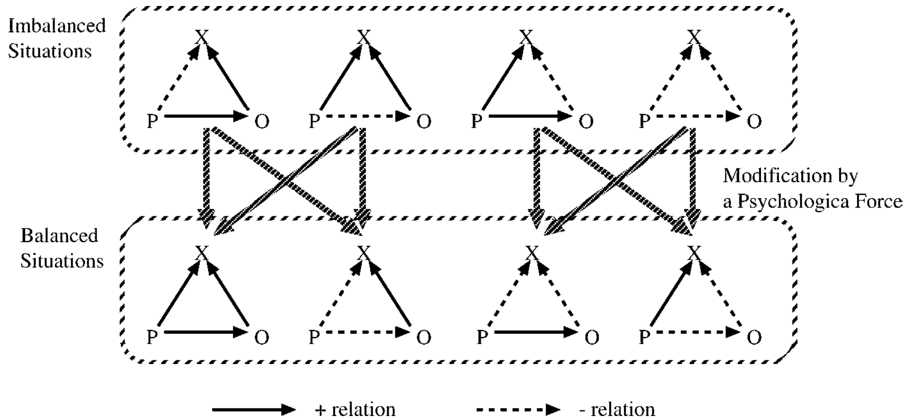

Fig. 1. Balanced and Imbalanced Situations of POX Systems in Heider's Balance Theory.

macro structure of group dynamics based on micro behaviors of the original POX systems in individual persons.

As an approach to this problem in the field of artificial societies, Wang and Thonegate [8] proposed a simulation model of group dynamics based on POX systems, consisting of full connected graphs. However, this study focuses on non–digraphs, that is, cases where all the dyad relations are symmetric. If more realistic situations should be considered, we need to analyze and simulate group dynamics of social networks represented as digraphs.

In this paper, we propose a formalization of group dynamics based on POX systems as a finite Markov chain with a state space consisting of signs on all the edges in digraphs, characterize the concept of balance as absorbing states of this Markov chain, and execute computer simulations of the group dynamics based on the Markov chain.

2 Group Dynamics based on POX Systems as a Finite Markov Chain

We assume that there are N persons and relations between them, and these relations have $+$ or $-$ value, where $+$ and $-$ mean that the person likes and dislikes the other person, respectively. Here, we do not deal with ambivalent states in individuals that have both $+$ and $-$ at the same time, or no sign. This social network can be represented as a signed digraph $G = (P, A)$, $P = \{p_1, p_2, \ldots, p_N\}$, $A = \{(a_1, s_1), (a_2, s_2), \ldots, (a_M, s_M)\}$. P is the set of N vertices in the digraph, corresponding to the N persons, and A is the set of pairs of edges $a_.$ and signs on them $s_.$.

If some persons change some signs on edges from them to others based on balance of their POX systems mentioned in the previous section, the vector

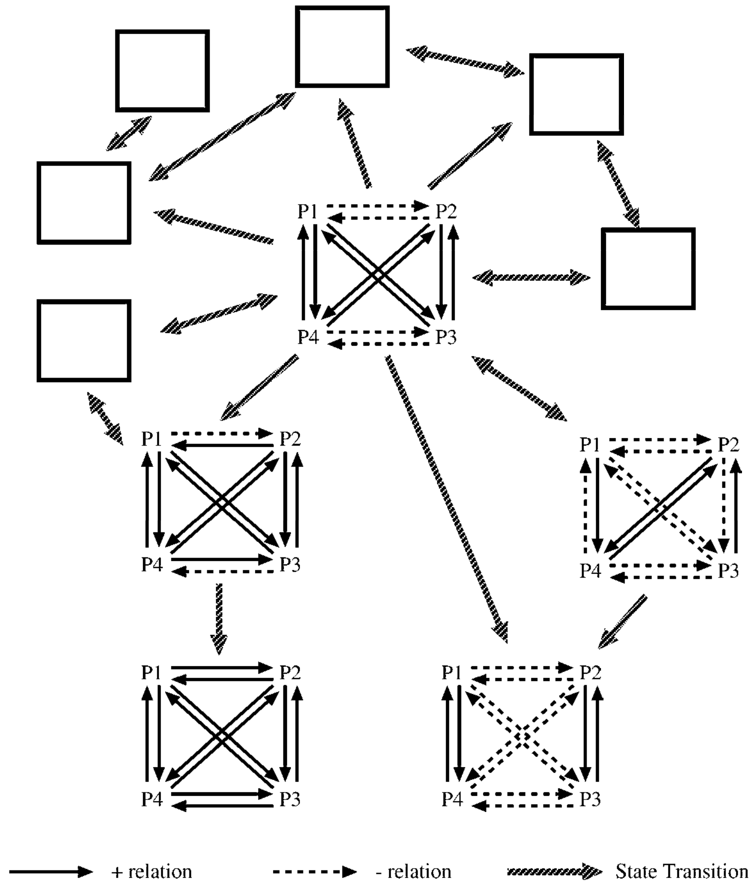

Fig. 2. An Example of the Finite Markov Chain of the Group Dynamics (4 Persons)

of the signs $(s_1, s_2, \ldots, s_M)$ is modified without changing the vertex and edge structures. Since selection of persons and their edges in this modification is stochastic and dependent only on the current signs of relations, group dynamics based on this modification process equals to a finite Markov chain with the state space $S = \{(s_1, s_2, \ldots, s_M) : \; s_i = +1, -1\}$, in which the total number of states is 2^M. Fig. 2 shows an example of this finite Markov chain with 4 persons who have relations each other, that is, on a full connected digraph with 4 vertices.

2.1 Relations between Absorbing States and Balanced Situations

Absorbing states in the above finite Markov chain as group dynamics are situations where all the POX systems are balanced. Some conventional and mathematical results clarify characteristics of these states.[1]

[1] In this paper, we refer to Hiramatsu [6] to mention these mathematical studies.

In cases of general digraphs and graphs, that is, social networks that have more than three persons and are not necessarily full connected ones, Cartwright and Harary [1] defined balanced situations as those where all the cycles (semi–cycles in cases of digraphs) in the networks are positive, that is, for any cycle (semi–cycle) the product of signs on all the edges in the cycle is positive.

In cases of both digraphs and graphs, Harary et. al., [3] proved a **structure theorem** arguing that a situation is balanced if and only if the set of vertices can be partitioned into two subsets so that all the edges between vertices in the same subgroup are signed with + and all the edges between vertices in different subgroups are signed with − (it is permitted that one of the subgroups is empty).

Furthermore, Flament [2] proved that balanced situations mentioned above equal to situations where all the triangles are positive. Since triangles are necessarily not POX systems (e.g., $A \to B \to C \to A$ is a triangle, but not a POX system), the condition that all the triangles are positive is stronger than the condition that all the POX systems are balanced.

If a graph is a fully connected digraph, that is, a digraph in which there is necessarily an edge between all the pairs of vertices, we can show that the following three conditions are equal each other:

1. A fully connected digraph is balanced in the sense of Cartwright and Harary's definition, that is, all the semi–cycles in the digraph are positive.
2. All the POX systems in the digraph are balanced.
3. The set of vertices in the digraph can be partitioned into two subsets so that all the edges between vertices in the same subgroup are signed with + and all the edges between vertices in different subgroups are signed with −.

$1 \Rightarrow 2$ is trivial since any POX system is a semi–cycle.

$2 \Rightarrow 3$ can be proved as follows.

If all the POX systems are balanced, all the dyad relations are symmetric since an asymmetric dyad relation between p_i and p_j causes imbalance in POX systems of the third person p_k. Then, the following relation between vertices is an equivalence relation:

$$p_i \sim p_j \stackrel{\text{def}}{=} \{p_i = p_j \text{ or the edge } p_i \to p_j \text{ is } +\}$$

For this equivalence relation $\sim$, vertices in the same equivalence class are connected each other by edges with + and vertices in the different classes are connected each other by edges with −. If there are more than two classes, POX systems consisting of vertices in the different three classes are imbalanced. Thus, there are at most two classes, and this situation equals to the statement of 3.

$3 \Rightarrow 1$ is trivial since every semi–cycle crossing the two subsets has an even number of crossings between the subsets.

The above mathematical result means that absorbing states of the finite Markov chain equal to balanced situations in the conventional sense. In other words, stable states of group dynamics based on micro behaviors of POX systems are situations where all the persons like each other or the group is separated into two subgroups in which all the persons in the same subgroup like each other and the persons in the different groups dislike each other.

The above proof for the relation between balanced stuations and absorbing states of the Markov chain based on POX systems can simply be applicable for cases of non–digraphs. Wang and Thonegate [8] showed based Monte Carlo simulations that group dynamics of POX systems converged to balanced situations in cases of fully connected non–digraphs (including both cases with and without ambivalent states), and our mathematical analysis supports their result.

2.2 Influence of a Fixing Person

As shown in the previous section, group dynamics of N persons having relations each other based on POX systems is a finite Markov chain of which absorbing states are situations where all the persons like each other or the persons are partitioned into two subgroups disliking each other. Since there are $N(N-1)$ relations in this case, the finite Markov chain has a total of $2^{N(N-1)}$ states. Moreover, the total number of the absorbing states is $\frac{1}{2}\sum_{i=0}^{N} C_i^N = 2^{N-1}$.

Here, we consider another finite Markov chain by adding the $(N+1)$-th person p_{N+1} to the original N persons. We assume that this person does not change relations to the other persons through group dynamics. In this paper, we call this person "a fixing person".

We represent a state of the original finite Markov chain at a time t as the following $N \times N$ matrix $MS(t)$:

$$MS(t) = \begin{pmatrix} - & s_{12}(t) & s_{13}(t) & \cdots & s_{1N}(t) \\ s_{21}(t) & - & s_{23}(t) & \cdots & s_{2N}(t) \\ s_{31}(t) & s_{32}(t) & - & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & s_{(N-1)N}(t) \\ s_{N1}(t) & s_{N2}(t) & \cdots & s_{N(N-1)}(t) & - \end{pmatrix} \quad (1)$$

Here, $s_{ij}(t)$ is the relation from the i-th person p_i to the j-th person p_j at the time t, and all the diagonal elements in MS are empty. Then, we can represent a state of the newly constructed finite Markov chain at the time t as the following $(N+1) \times (N+1)$ matrix $NS(t)$:

$$NS(t) = \begin{pmatrix} & & & s_{1(N+1)}(t) \\ & MS(t) & & \vdots \\ & & & s_{N(N+1)}(t) \\ s_{(N+1)1}(t) & \cdots & s_{(N+1)N}(t) & - \end{pmatrix} \quad (2)$$

Here, all the elements $s_{(N+1)j}(t)$ $(j = 1, \ldots, N)$ in the $(N+1)$-th row are fixed through the process, that is, $s_{(N+1)j}(t) = s_{(N+1)j}(0)$ $(t \geq 1)$. Thus, the newly constructed Markov chain has a total of 2^{N^2} states.

We consider absorbing states of the above finite Markov chain. If p_{N+1} has $+$ relations to p_{i_1}, …, p_{i_d} and $-$ relations to $p_{i_{d+1}}$, …, p_{i_N}, and all the POX systems of p_1, ..., p_N including relations to p_{N+1} are balanced, the group is partitioned into two subgroups $A = \{p_{i_1}, \ldots, p_{i_d}, p_{N+1}\}$ and $B = \{p_{i_{d+1}}, \ldots, p_{i_N}\}$, all the persons in the same subgroup have + relations each other, and all the persons in the different subgroups have − relations each other. In fact, $\{p_1, \ldots, p_N\}$ is partitioned into two subgroups A' and B' disliking each other by the characteristic shown in the previous section. If $p_i, p_j \in A'$ and $p_k \in B'$, balance of all the POX systems implies that $s_{i(N+1)} = s_{j(N+1)} \neq s_{k(N+1)}$. Furthermore, it implies that $s_{(N+1)i} = s_{i(N+1)} \neq s_{k(N+1)} = s_{(N+1)k}$. These facts imply that $A = A' \cup \{p_{N+1}\}, B = B'$ or $A = B' \cup \{p_{N+1}\}, B = A'$.

This mathematical property means that a fixing person in the above sense influences the process of the finite Markov chain so that there is only one absorbing state where the grouping intended by the fixing person is realized. Particular cases of $d = N$ and $d = 0$ correspond to the situation where all the persons including p_{N+1} like each other, and the situation where all the persons except for p_{N+1} like each other and p_{N+1} is disliked by them, respectively.

The existence of a fixing person suggests that it can operate structures of the group to some extent. For example, there is a folk psychological discourse that if one of members in a group plays a role of a villain exclusive for other members they are settled. In fact, a family psychologist interprets phenomena of juvenile delinquents having parents on bad terms as balance of the family POX system in the same sense as our model with a fixing person [4]. The case of $d = 0$ in our model with a fixing person gives a mathematical meanings to this discourse.

3 Simulations

As shown in the previous section, the group dynamics based on individual POX systems is represented as a finite Markov chain having absorbing states corresponding to situations where all the person like each other, or the persons are partitioned into two subgroups disliking each other. In addition, the existence of a person who fixes all the relations to the others modifies the structure of the original Markov chain to limit absorbing states to only one state where the grouping intended by the fixing person is realized. However, this analysis does not clarify whether this finite Markov chain has cyclic states. In other words, there is a possibility of the existence of cyclic states where modification of some POX systems and that of other POX systems are repeated one another.

In case of symmetric dyad relations, that is, non–directed social networks, Wang and Thonegate suggested based on computer simulations that there is no such cyclic state [8]. In this paper, we also verify based on computer simulations what states appear through the above finite Markov chain of group dynamics based on POX systems.

We execute our simulations based on the following procedures:

1. Initialize $\{s_{ij}(0)\}$ in equations (1) and (2) into +1 or −1 randomly.
2. Execute the following procedures synchronously for $i = 1, \ldots, N$:
   (a) Select j and k randomly $(i \neq j, i \neq k, j \neq k)$.

(b) If $s_{jk}(t)s_{ij}(t)s_{ik}(t) = -1$, then $s_{ij}(t+1) = -s_{ij}(t)$, $s_{ik}(t+1) = s_{ik}(t)$, or $s_{ij}(t+1) = s_{ij}(t)$, $s_{ik}(t+1) = -s_{ik}(t)$
($s_{il}(t+1) = s_{il}(t)$ for $l \neq j, k$)

3. Iterate from $t = 1$ to $t = T - 1$ or until $s_{jk}(t)s_{ij}(t)s_{ik}(t) = +1$ is satisfied for any triplet (i, j, k) $(i \neq j, i \neq k, j \neq k)$.

We set $T = 10^7$ and run 300 trials with different random seeds for one simulation configuration.

3.1 Cases without a Fixing Person

We first tried 5 configurations: 4 persons, 5 persons, 6 persons, 7 persons, and 8 persons without a fixing person. Table 1 shows types of grouping in absorbing states and the numbers of the corresponding states, the numbers of trials that converged to the corresponding states and their rates among 300 trials, average number of iteration for convergence to each grouping, and its standard deviation for each configuration.

As shown in Table 1, the state converged to one of absorbing states shown in the previous section in all the trials for all the configurations, and any cyclic state was not observed. However, there is almost no difference between the numbers of trials for convergence to the absorbing states. For example, in the case of 4 persons, there are 4 states with $(3 : 1)$ grouping and 3 states with $(2 : 2)$ grouping, and the average number of trials for convergence to each state is 41.1 in the case of $(3 : 1)$ and 35.3 in the case of $(2 : 2)$. Since the number of trials for convergence to $(4 : 0)$ (the situation where all the persons like each other) is 29, no trend existed that there is a specific absorbing state where convergence is concentrated. Moreover, no trend existed that there is a specific absorbing state to which convergence is faster than the other states. The same fact is shown in the other configuration.

3.2 Cases with a Fixing Person

Next, we tried another 4 configurations: 5 persons including a fixing person, 6 persons including a fixing person, and 7 persons including a fixing person, and 8 persons including a fixing person. We assume that the fixing person has fixed $-$ relations to all the other persons ($d = 0$). Table 2 shows the number of trials that converged to the absorbing state intended by the fixing person (the situation where all the persons except for the fixing person like each other and the fixing person is disliked by them) among 300 trials, average number of iteration for convergence to the state, and its standard deviation for each configuration.

As shown in Table 2, the state converged to only one absorbing state in all the trials for all the configurations, and any cyclic state was not observed.

3.3 Numbers of Iteration Required for Convergence

As mentioned in the previous section, as the number of persons increases from N to $N + 1$, the number of states of the Markov chain without a fixing person

Table 1. Results of the Simulations without a Fixing Person for 4–8 Persons (SD: Standard Deviation)

N	4		
Types of Grouping (#. Corresponding States)	4 : 0 (1)	3 : 1 (4)	2 : 2 (3)
#. Trials that Converged to the States (Rates in the Total Number)	29 (9.7%)	165 (55%)	106 (35.3%)
Average #. Iteration for Convergence (SD)	37.6 (30.4)	37.1 (35.5)	43.2 (42.4)

N	5		
Types of Grouping (#. Corresponding States)	5 : 0 (1)	4 : 1 (5)	3 : 2 (10)
#. Trials that Converged to the States (Rates in the Total Number)	16 (5.3%)	108 (36%)	176 (58.7%)
Average #. Iteration for Convergence (SD)	185.4 (188.5)	184.9 (207.6)	160.9 (165.5)

N	6			
Types of Grouping (#. Corresponding States)	6 : 0 (1)	5 : 1 (6)	4 : 2 (15)	3 : 3 (20)
#. Trials that Converged to the States (Rates in the Total Number)	7 (2.3%)	63 (21%)	140 (46.7%)	90 (30%)
Average #. Iteration for Convergence (SD)	1069.1 (1209.2)	1022.1 (937.4)	1220.8 (1161.3)	1340.3 (1351.5)

N	7			
Types of Grouping (#. Corresponding States)	7 : 0 (1)	6 : 1 (7)	5 : 2 (21)	4 : 3 (35)
#. Trials that Converged to the States (Rates in the Total Number)	5 (1.7%)	29 (9.7%)	91 (30.3%)	175 (58.3%)
Average #. Iteration for Convergence (SD)	15129.2 (10085.8)	11847.8 (10660.2)	13135.3 (13374.7)	15224.4 (13430.9)

N	8		
Types of Grouping (#. Corresponding States)	8 : 0 (1)	7 : 1 (8)	6 : 2 (28)
#. Trials that Converged to the States (Rates in the Total Number)	2 (0.7%)	14 (4.7%)	62 (20.7%)
Average #. Iteration for Convergence (SD)	241338.5 (51426.5)	194909.1 (281313.9)	239980.8 (259982.4)

N	8	
Types of Grouping (#. Corresponding States)	5 : 3 (56)	4 : 4 (35)
#. Trials that Converged to the States (Rates in the Total Number)	148 (49.3%)	74 (24.7%)
Average #. Iteration for Convergence (SD)	259982.8 (256895.0)	224161.2 (219238.3)

Table 2. Results of the Simulations with a Fixing Person for 5–8 Persons (SD: Standard Deviation)

#. Persons	5	6	7	8
#. Trials that Converged to the State Intended by the Fixing Person	300	300	300	300
Average #. Iteration for Convergence to the State (SD)	666.6 (627.8)	4719.9 (4762.9)	60699.0 (62077.6)	1102131.3 (1136276.6)

increases from $2^{N(N-1)}$ to $2^{N(N+1)}$, that is, 2^{2N} times. That with a fixing person increases from $2^{(N-1)^2}$ to 2^{N^2}, that is, 2^{2N-1} times. Here, we investigate the influence of this increment and the existence of a fixing person to the number of iteration needed for convergence to the absorbing states.

Fig. 3 shows the average numbers of interation needed for convergence to the absorbing states in the simulations shown in section 3.1 and 3.2. Note that the vertical axis in the figure is shown with log scale. This figure suggests that the number of iteration needed for convergence to the absorbing states almost exponentially increases as the number of persons increases, in both cases with a fixing person and without a fixing person.

Moreover, Fig. 3 also suggests that the average numbers of interation for convergence in cases with a fixing person are a constant times as large as those in cases without a fixing person. The numbers of states in cases of N persons including a fixing person and not including are $2^{(N-1)^2}$ and $2^{N(N-1)}$ respectively. The decrement rate of numbers of states by the existence of a fixing person is $2^{-(N-1)}$. Moreover, the numbers of absorbing states in cases of N persons including a fixing person and not including are 1 and 2^{N-1} respectively. The decrement rate of numbers of absorbing states by the existence of a fixing person is also $2^{-(N-1)}$, and thus the rate of absorbing states in all the states is $2^{-(N-1)^2}$ in both cases. Nevertheless, the Markov chain with a fixing person spent much more time than that without a fixing person for convergence.

4 Conclusion and Discussion

In this paper, we proposed a formalization of group dynamics based on balance of individual POX systems as a finite Markov chain, analyzed relations between absorbing states of this Markov chain and balanced situations, and verified the mathematical characteristics by executing computer simulations. Moreover, we mathematically analyzed and executed computer simulations to verify influences of a specific person who fixes relations to others through process, a fixing person.

In both the case without a fixing person and that with a fixing person, no cyclic state was observed in the simulation results of the finite Markov chain. It suggests that as far as persons behave only based on balance theory the situation

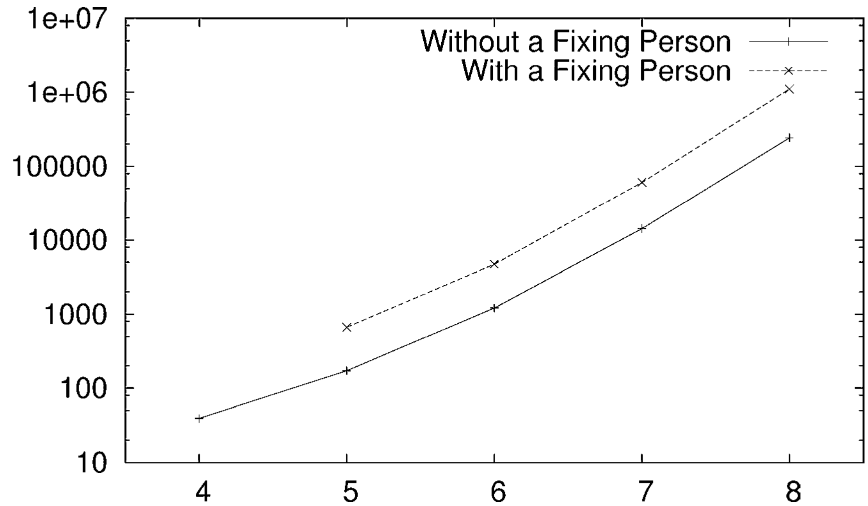

Fig. 3. The Average Numbers of Interation Needed for Convergence to the Absorbing States in the Simulations (the horizontal axis: the numver of the persons N)

of the group converges to a balanced situation. If all the persons modify their POX systems based on their balance, there is a trend that the group is separated into two subgroups exclusive each other since the situation where all the persons like each other is just one possibility among many absorbing states.

Moreover, the number of iteration needed for convergence to the absorbing states extremely increase as the number of persons increases. In particular, more than 200,000 iterations were needed for convergece for the case of 8 persons. On the other hand, Wang and Thonegate [8] reported that at most 100,000 iterations were needed for convergence even in cases of 25 persons in their Monte Carlo simulations. Although we cannot naively compare our results to those, this fact suggests that convergence to balanced situations in a larger group spends much more time in more realistic situations.

Moreover, a fixing person reduces the number of absorbing states in the original Markov chain in the sense that the state where the grouping intended by the fixing person is realized becomes only one absorbing state. On the other hand, convergence to the only absorbing state need more time than convergence in the original Markov chain. This fact suggests that it is hard to control the grouping based on relations of POX systems in more realistic situations.

However, our formalization in this paper have some problems.

First, influence of a fixing person in our model lacks a realistic meanings. In our model, the fixing person can influence to the group so that any grouping is realized. In particular, the case of $d = N$ means that the fixing person can lead to the situation where all the persons including itself like each other. From fork psychological perspectives, however, it seems to be less possible than the case that the fixing person plays as a villain, for example, as in case of juvenile delinquents having parents on bad terms [4].

We should consider that real social networks are influenced by many factors like individual experiences and cognitive properties, in particular, cognitive bias for information with negative meanings. As one of future problems, we should consider social network models including artificial agents with these individual and cognitive properties. Moreover, we should extend our formalization to general digraphs that are not fully connected.

Moreover, we confirmed only based on computer simulations that no cyclic states were not observed. Finally, it should analytically proved, for example, by classifying states of the Markov chain based on locality of balanced POX systems in the whole group.

Acknowledgments

This study was supported by the Japan Society for the Promotion of Science, Grants–in–Aid for Scientific Research No. 15330133. Moreover, we thank Dr. Takashi Fujisawa in Kansai University for his support on investigation of the conventional mathematical research results related to this study.

References

1. Cartwright, D. and Harary, F. 1956. Structural balance: A generalization of Heider's theory. *Psychological Review* 63: 277.
2. Flament, C. 1963. *Applications of Graph Theory to Group Structure.* Printice–Hall. (Japanese translation: Yamamoto, K. 1974. Kinokuniya).
3. Harary, F., Norman, R. Z., and Cartwright, D. 1965. *Structural Models: An Introduction to the Theory of Directed Graphs.* John Wiley & Sons.
4. Hasegawa, K. 1987. *Paradox in Families.* Seiko–Shobo. (in Japanese).
5. Heider, F. 1958. *The Psychology of Interpersonal Relations.* John Wiley & Sons. (Japanese translation: Ohashi, M. 1978. Seishin Shobo).
6. Hiramatsu, H. 1990. *Social Networks.* Fukumura–Shuppan. (in Japanese).
7. Roberts, F. S. 2000. On balanced signed graphs and consistent marked graphs. In *Electric Notes in Discrete Mathematics* 2, edited by Melvin F. Janowitz. Elsevier. (electric publication: `http://www1.elsevier.com/gej-ng/31/29/24/32/23/89/disc2008.ps`
8. Wang, Z. and Thorngate, W. 2003. Sentiment and social mitosis: Implications of Heider's Balance Theory. *J. Artificial Societies and Social Simulations* 6 3. (electric jounal: `http://jasss.soc.surrey.ac.uk/6/3/2.html`).